Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(99)00014-3

J. Appl. Maths Mechs, Vol. 63, No. 1, pp. 93-100, 1999

0021-8928/99/$—see front matter

THE AVERAGED EQUATIONS OF THE DYNAMICS OF THIN MULTILAYERED PACKETS OF ARBITRARY STRUCTURE WITH CONTRASTING DIRECTIONS OF ANISOTROPY IN THE ELASTIC LAYERS†

D. D. ZAKHAROV

Moscow

(*Received* 25 *December* 1997)

Using the three-dimensional dynamical theory of elasticity, the asymptotic method [1, 2] is used to derive two-dimensional equations and elasticity relations for a packet with an arbitrary anisotropy and arrangement of the layers. It is assumed that the contact between layers is ideal and the packet is subject to stresses on the face surfaces with certain boundary conditions at the ends. It is also assumed that in addition to the usual small parameter $\varepsilon = h/L \ll 1$ for a long-wave approximation (h is the half-thickness and L is the scale of the process in a longitudinal direction), the ratios of the elastic moduli in different directions in each layer can generate additional small parameters of the form $\varepsilon^p, p > 0, \varepsilon \to +0$. Similar contrasting differences are permitted between groups of layers. The results are classified in terms of the contrast characteristics and types of anisotropy.

The main purpose of this paper is to obtain an asymptotically exact description of the dynamic internal stress-strain state (SSS) of a thin packet of anisotropic layers with contrasting properties. These include a large difference between the elastic properties in the different anisotropy directions within any layer (the existence of contrasting directions) as well as a large difference between layers or groups of layers.

The first property is typical of modern heavy-duty unidirectional composites, where the moduli of the carbon or boron-carbon fibres in a layer are substantially greater than the moduli of the matrix [3], resulting in complex properties of the packet as a whole. The many attempts that have been made to describe them mathematically reduce essentially to the use of high-order deformation theories which give rise to technical difficulties of their own [4-6]. In terms of asymptotic behaviour, it means that one must allow for an increasingly large number of terms (of the order of four, six or more) in the respective asymptotic series of standard form (as many as in the non-contrasting case). However it would seem natural to begin with a long-wave approximation, confining ourselves to one or two terms of the series, and then correcting them by introducing extra small parameters (such as the ratios of the elastic moduli in different directions). This is what we do here.

The second property is frequently found in panels of composite materials with outer supporting layers and a relatively soft filler [3, 7-11], so that the contrast manifests itself on going from layer to layer. The hypothesis method is widely used in the theory of laminated structures with a small physical parameter of this kind [7, 12, 13]. Asymptotic methods were developed for the case of isotropic or transversally-isotropic materials in [14].‡

1. PHYSICAL AND MATHEMATICAL STATEMENT OF THE PROBLEM

Let the jth layer occupy the region $\Omega \times [z_j, z_{j+1}]$ ($j = 1, 2, \ldots, N$) in a Cartesian system of coordinates $\mathbf{x} = \mathbf{i}_\alpha x_\alpha, x_3 = z$. We assume that the linearly elastic material of the layer possesses general anisotropy and the three-dimensional Hooke's law corresponds to a sixth-order stiffness matrix $\mathbf{G}_j$ containing 21 independent elastic constants. We will denote the density of the material by ρ_j.

The layers adhere perfectly to the contact surfaces and are arbitrarily and asymmetrically arranged in a packet.

For certain boundary conditions at the ends $\partial\Omega \times [z^-, z^+]$, $z_1 = z^-$, $z_{N+1} = z^+$, the packet is assumed to be subject to the effect of distributed stresses on the exposed surfaces, so that the characteristic scale

†*Prikl. Mat. Mekh.* Vol. 63, No. 1, pp. 102-110, 1999.

‡See also LOZKHIN, O. B., Some problems of the axisymmetric bending of three-layer shells of revolution. Candidate dissertation 01.02.03, Mosk. Aviats. Inst., Moscow, 1976.

L of the dynamic process in a longitudinal plane is much greater than the thickness of the packet $2h = h_1 + h_2 + \ldots + h_N$, $h_j = z_{j+1} - z_j$, that is, $\varepsilon = hL^{-1} \ll 1$ is a small geometrical parameter.

We will denote the characteristic values of the elastic modulus and density (the largest values in the packet, for instance) by E_0 and ρ_0.

We will further assume that the coordinates $\mathbf{x}$, z are normalized to L and h, and that the quantities $\mathbf{G}_j$ and ρ_j are normalized to E_0 and ρ_0, respectively. The dimensionless time t is obtained by normalizing to a certain characteristic scale $T = \varepsilon^{\tau-1} l c_0^{-1}$, $c_0^2 = E_0 \rho_0^{-1}$. The loads on the lateral surfaces are given in the form

$$\sigma_{33}^{\mp} = \sigma^{\mp}(\mathbf{x}, t), \quad \sigma_{\alpha 3}^{\mp} = \varepsilon^{-1} \tau_\alpha^{\mp}(\mathbf{x}, t) \quad (z = z^{\mp}).$$

For unitary subscripts of the stresses and strains the subscripts 1, 2, 3 in the stiffness matrix $\mathbf{G}_j$ below will correspond to the subscripts 11, 22, 33, and the subscripts 4, 5, 6 will correspond to the subscripts 23, 13, 12 for shear stresses and strains.

It will also be assumed for the stiffnesses that

$$G_{mn}^j = \varepsilon^p g_{mn}^j \quad (m = 3,4,5), \quad G_{mn}^j = \varepsilon^q q_{mn}^j \quad (n = 1,6,2),$$

where $g_{mn}^j = O(1)$, $\varepsilon \to +0$. The largest of the powers p and q $(p, q > 0)$ are chosen at the intersection of the mth rows and nth columns.

We will begin by confining ourselves to an investigation of p, $q = 1, 2, 3$. As a rule, this is adequate in thin bodies, $\varepsilon < 10^{-1}$, and for a range of real values of the parameters.

Thus, the displacements, strains and stresses in the layers satisfy the equations of the three-dimensional dynamic theory of elasticity, Hooke's law, the conditions for total adhesion of the layers, the specified loads on the faces and certain boundary conditions at the end of the packet.

For an internal SSS of the packet (far away from the ends) the displacements $\mathbf{U} = \mathbf{i}_\alpha U_\alpha$, $U_3 = W$, strains $\varepsilon_{\alpha\beta}$ and stresses $\Sigma_{\gamma\delta}$ will be sought in the form of asymptotic series in powers of ε (j, the number of the layer, is omitted in obvious causes)

$$\mathbf{U} = h\varepsilon^\lambda (\mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \ldots), \quad W = h\varepsilon^\mu (w^0 + \varepsilon w^1 + \ldots)$$
$$\varepsilon_{\gamma\delta} = \varepsilon^\theta (\varepsilon_{\gamma\delta}^0 + \varepsilon \varepsilon_{\gamma\delta}^1 + \ldots), \quad \Sigma_{\gamma\delta} = E_0 \varepsilon^\kappa (\sigma_{\gamma\delta}^0 + \varepsilon \sigma_{\gamma\delta}^1 + \ldots)$$

with the different superscripts λ, μ, θ, κ, τ corresponding to the choice of the (p, q)-model of the stiffnesses. We obtain the following orders and recurrence relations for the dimensionless quantities

$$\gamma_{\alpha\beta}^s = (\partial_\alpha u_\beta + \partial_\beta u_\alpha)^s (\theta = \lambda + 1), \quad \varepsilon_{zz}^s = \partial_z w^s (\theta = \mu) \tag{1.1}$$
$$\gamma_{\beta z}^s = \partial_z u_\beta^{s+\theta-\lambda} + \partial_\beta w^{s+\theta-\mu-1} (\theta = \min(\lambda, \mu + 1))$$
$$\sigma_\delta^s = (g_{\delta 1}\varepsilon_{11} + g_{\delta 6}\gamma_{12} + g_{\delta 2}\varepsilon_{22})^{s+\kappa-q-\lambda-1} + g_{\delta 3}\varepsilon_{zz}^{s+\kappa-p-\mu} + (g_{\delta 4}\gamma_{2z} + g_{\delta 5}\gamma_{1z})^{s+\kappa-p-\xi} \tag{1.2}$$
$$(\kappa = \min(q + \lambda + 1, p + \mu, p + \xi); \ \delta = 1, 2, \ldots, 6)$$
$$\partial_\beta \sigma_{\alpha\beta}^{s+\lambda_0-\kappa-1} \partial_z \sigma_{\alpha z}^{s+\lambda_0-\kappa} = \rho_j \partial_t^2 u^{s+\lambda_0+2\tau-4-\lambda}$$
$$\partial_\beta \sigma_{\beta z}^{s+\mu_0-\kappa-1} \partial_z \sigma_{\alpha z}^{s+\mu_0-\kappa} = \rho_j \partial_t^2 w^{s+\mu_0+2\tau-4-\mu} \tag{1.3}$$
$$(\lambda_0 \equiv \min(\lambda, \kappa), \ \mu_0 \equiv \min(\mu, \kappa))$$

The elasticity relations (1.1), (1.2) and Eqs (1.3) are supplemented by the conditions on the contact surfaces between layers and the faces

$$z = z_{j+1} : \sigma_{\alpha z j}^{s-\kappa_j} = \sigma_{\alpha z j+1}^{s-\kappa_{j+1}}, \quad \sigma_{zzj}^{s-\kappa_j} = \sigma_{zzj+1}^{s-\kappa_{j+1}}$$
$$u_{\alpha j}^{s-\lambda_j} = u_{\alpha j+1}^{s-\lambda_{j+1}}, \quad w_j^{s-\mu_j} = w_{j+1}^{s-\mu_{j+1}} \tag{1.4}$$
$$z = z^{\mp} : \sigma_{\alpha z \mp}^s = \tau_\alpha^{\mp} \delta_{s+\kappa_\mp}^{-1}, \quad \sigma_{zz\mp}^s = \sigma^{\mp} \delta_{s+\kappa_\mp}^0$$

where the superscripts $\mp$ and $1, N+1$ are identified for brevity; $j = 2, 3, \ldots, N-1$, and δ_n^m is the Kronecker delta.

2. THE BASIS MODEL OF A PACKET OF (0, 0)-LAYERS

This model was studied in detail [16–18], but it is interesting to take it as a basis for comparison with other models. We will give the main expressions and orders for the principal terms ($s = 0$)

$$\tau = 0,\ \mu = -4,\ \lambda = -3$$

$$\mathbf{u}^0 = \mathbf{u}(\mathbf{x},t) - z\nabla w,\ w^0 = w(\mathbf{x},t),\ \nabla = \mathbf{i}_\alpha \partial_\alpha \tag{2.1}$$

$$\sigma^j_{\alpha\beta} = \chi_{\alpha\beta}(\mathbf{\Gamma}_j)\mathbf{u}^0,\ \kappa = -2\ (\alpha\beta = 11, 12, 22) \tag{2.2}$$

$$\chi_{11}(\Gamma) \equiv (\gamma_{11}\partial_1 + \gamma_{16}\partial_2)\mathbf{i}_1 + (\gamma_{16}\partial_1 + \gamma_{12}\partial_2)\mathbf{i}_2\ (1 \leftrightarrow 2)$$

$$\chi_{12}(\Gamma) \equiv (\gamma_{16}\partial_1 + \gamma_{66}\partial_2)\mathbf{i}_1 + (\gamma_{66}\partial_1 + \gamma_{26}\partial_2)\mathbf{i}_2$$

$$\sigma^j_{\alpha z} = \tau^\pm_\alpha \pm \Sigma_\pm h_l \mathbf{a}^l_\alpha(\mathbf{u} - z^0_l\nabla w) + $$

$$+\ (z^\mp_j - z)\mathbf{a}^j_\alpha\left(\mathbf{u} - \frac{(z + z^\mp_j)}{2}\nabla w\right),\ \kappa = -1 \tag{2.3}$$

$$\sigma^j_{zz} = \sigma^\pm + (z^\pm - z)\nabla\mathbf{\tau}^\pm + [(z - z^\pm_j)\rho_j \mp \Sigma_\pm h_l\rho_l]\partial^2_t w \mp$$

$$\mp\ \Sigma_\pm\left[h_l(z - z^0_l)\mathbf{a}^l_*\mathbf{u} + \left(\frac{z^3_{l+1} - z^3_l}{2} - zh_l z^0_l\right)b^l_* w\right] +$$

$$+\ \frac{(z^\mp_j - z)^2}{2}\mathbf{a}^j_*\mathbf{u} - \left[\frac{z^3}{6} - \frac{z(z^\mp_j)^2}{2} + \frac{(z^\mp_j)^3}{3}\right]b^j_* w,\ \ \kappa = 0 \tag{2.4}$$

where the following notation has been used (N will also be used to denote the entire set of indices)

$$\mathbf{a}_\alpha = \partial_\beta \chi_{\alpha\beta},\ \mathbf{a}_* = \partial_\alpha \mathbf{a}_\alpha = \mathbf{i}_\alpha b_\alpha,\ b_* = \partial_\alpha b_\alpha = \mathbf{a}_\alpha \nabla$$

$$z^\mp_j = z_j, z_{j+1};\ 2z^0_j = z_j + z_{j+1}, \tag{2.5}$$

$$\Sigma_\mp = \sum_l\ (l < j\ \text{ or }\ l > j : l, j \in N)$$

$$\mathbf{\Gamma}_j = \|\gamma_{mn}\|_j,\ \gamma_{mn} = \det\mathbf{G}^m_n / \det\mathbf{G}_0\ \ (m, n = 1, 6, 2) \tag{2.6}$$

In the elements of the matrix Γ_j of the averaged stiffnesses of the jth layer, $\mathbf{G}_0$ denotes the diagonal minor 3, 4, 5 in the matrix G_j, and G^m_n is the bordering minor obtained by adding the mth row and nth column to $\mathbf{G}_0$. In the special case of monoclinic anisotropy (relative to the exposed surfaces) in the layer the stiffness formulae become simpler: $\gamma_{mn} = g_{mn} - g_{m3}g_{3n}/g_{33}$.

The displacements are independent of the layer number and satisfy the system of equations

$$\mathbf{a}_\alpha(\mathbf{D}_1)\mathbf{u} - b_\alpha(\mathbf{D}_2)w + \tau^+_\alpha - \tau^-_\alpha = 0$$

$$-\mathbf{a}_*(\mathbf{D}_2)\mathbf{u} + [\rho_*\partial^2_t + b_*(\mathbf{D}_3)]w = \sigma^+ - \sigma^- + \nabla(z^+\mathbf{\tau}^+ - z^-\mathbf{\tau}^-) \tag{2.7}$$

$$\mathbf{D}_k = \sum_j \int_{z_j}^{z_{j+1}} z^{k-1}\mathbf{\Gamma}_j dz,\ \rho_* = \sum_j \int_{z_j}^{z_{j+1}} \rho_j dz \tag{2.8}$$

When the layers are arranged asymmetrically, the classical bending ($\mathbf{D}_3$) and membrane ($\mathbf{D}_1$) stiffnesses are supplemented by mixed membrane-bending integral stiffnesses of the packet ($\mathbf{D}_2 \neq 0$). Note that the existence of monoclinic anisotropy has no effect on the kinematic relations and other equations for first-order terms ($s = 1$) (apart from homogeneous equations (2.7)), but total anisotropy results in a violation of Kirchhoff's relations for them (the effect of transverse shear strains [18]).

3. A PACKET OF $(p,\ 0)$-LAYERS

In this case the materials are stiffer in the longitudinal direction and relatively soft in the transverse direction. When $p = 1$ the relations for the principal terms $s = 0$ of Eqs (1.1)–(1.4) are the same as in the (0, 0)-model, but the construction of the matrix of averaged stiffnesses in the jth layer is different. For $\mathbf{F}_j$ we obtain the series

$$\mathbf{\Gamma}_j = \mathbf{\Gamma}_j^* + \varepsilon \mathbf{\Gamma}_j^{**} + \dots \tag{3.1}$$

and must replace $\mathbf{\Gamma}_j$ in Eqs (2.2)–(2.8) by $\mathbf{\Gamma}_j^*$. Here

$$\gamma_{mn}^j = g_{mn}^j \tag{3.2}$$

Even though there are now short and slow transverse waves, the characteristic timescale corresponding to $\tau = 0$ is unchanged.

Note that even when $s = 0$ the case $p \geqslant 2$ yields a system of equations with inseparable coordinates $\mathbf{x}$ and z, i.e. the recurrence in relations (1.1)–(1.4) is lost.

Note. The loss of recurrence is an effect of the transverse shear strains. For example, the situation is not improved by putting

$$G_{mn} = O(\varepsilon^p) \ \ (m = 4,5), \ \ G_{3n} = O(1) \tag{3.3}$$

Nor is it any better with monoclinic anisotropy or orthotropy. But relations (2.2)–(2.8) do not change at all if we confine ourselves to the case of high compliance to transverse tension-compression

$$G_{mn} = O(1) \ \ (m = 4,5), \ \ G_{3n} = O(\varepsilon^p) \tag{3.4}$$

and all we need do is select the appropriate matrix Γ_j^* from (3.1).

We will now consider the kinematic relations for $s = 1$. If $p = 1$ we have

$$w^1 = w^1(\mathbf{x},t), \ \ \mathbf{u}^1 = \mathbf{u}_0^1(\mathbf{x},t) - z\nabla w^1 + \mathbf{\Delta}' \int_0^z \boldsymbol{\tau}_*^0 - \mathbf{G}_*'\boldsymbol{\gamma}^0 dz \tag{3.5}$$

$$\mathbf{\Delta} = \mathbf{G}_0^{-1}, \ \ \mathbf{G}_* = \mathbf{G}\binom{345}{162}, \ \ \boldsymbol{\gamma} = (\varepsilon_{11}, \gamma_{12}, \varepsilon_{22})^T, \ \ \boldsymbol{\tau}_* = (0, \sigma_{1z}, \sigma_{2z})^T$$

where Δ' and $\mathbf{G}_*'$ denote the matrices Δ and $\mathbf{G}_*$ with the first rows deleted and the remaining rows transposed. In the case of monoclinic anisotropy $\mathbf{G}_* = 0$ and $\Delta_{12} = \Delta_{13} = 0$. The deflection is clearly the same in all layers and the longitudinal displacements depend on the number of the layer. The longitudinal shear strains are never zero, so that to a first approximation this always differs from Kirchhoff's kinematic model.

For contrasting stiffnesses (3.4), $p = 1, 2$ one must put $\tau_* = 0$ in formula (3.5) and replace $\mathbf{\Delta}'$ by the matrix $\mathbf{\Delta}''$ (obtained by deleting the first column as well). We again obtain Kirchhoff's kinematic relations in the case of monoclinic anisotropy.

In case (3.4) $p = 3$ the relations for $s = 1$ change and the transverse tension–compression stresses make a substantial contribution

$$w^1 = w_0^1(\mathbf{x},t) + g_{33}^{-1} \int_0^z \sigma_{zz}^0 dz, \ \ \mathbf{u}^1 = \mathbf{u}_0^1(\mathbf{x},t) - \int_0^z \nabla w^1 + \mathbf{\Delta}''\mathbf{G}_*'\boldsymbol{\gamma}^0 dz$$

The simpler anisotropic case has kinematic relations of the same type.

Finally, the case (3.3) and $p = 1$ yields relations of the form (3.5) if again we use the matrix $\mathbf{\Delta}''$, the vector $\boldsymbol{\tau}_* = (\sigma_{13}, \sigma_{23})^T$, and replace $\mathbf{G}_*'$ by the matrix

$$\mathbf{G}_*^* = \mathbf{G}_*' - g_{33}^{-1} \left\| \begin{matrix} g_{53} \\ g_{43} \end{matrix} \right\| (g_{31}, g_{36}, g_{32}).$$

The simplifications to monoclinic anisotropy gives $\mathbf{G}_*^* = 0$, but leaves the dependence on the shear stresses τ_*.

4. A PACKET OF $(0, q)$-LAYERS

This corresponds to considerably greater compliance of the layers in the longitudinal than the transverse direction. Equations (2.1)–(2.8) hold for the principal terms $s = 0$ for the following values

$$\tau = -g/2, \quad \mu = -4-q, \quad \lambda = -3-q$$

with averaged stiffnesses (3.2) in the layers. There is no change in the order of the stresses.

Kirchhoff's kinematic relations are preserved up to the term of order q in the asymptotic series of the displacements, and up to the $(q + 1)$th term in the case of monoclinic anisotropy.

All the relations (2.1)–(2.8) still hold for the sum of terms $s = 0.1, \ldots, q\ (q + 1)$ with limiting error $O(\varepsilon^{q+1})(O(\varepsilon^{q+12}))$ if the complete stiffness matrix Γ of formula (3.1) is used in those equations.

5. A PACKET OF (p, q)-LAYERS

For a packet of $(q + r, q)$ layers with $r = 1$ if $p - q = r > 0$ we obtain

$$\tau = -q/2, \quad \mu = -4-q, \quad \lambda = -3-q$$
$$\kappa = -2-q\ (11, 12, 22), \quad \kappa = -1-q\ (13, 23), \quad \kappa = -q\ (33)$$

with the same relations (2.1)–(2.8) for the principal terms $s = 0$.

If $r \geqslant 2$, because of the loss of recurrence the variables $\mathbf{x}$ and z cannot be separated as in the case of $(r, 0)$-layers.

The model for a packet of $(p, p + r)$-layers with $q - p = r > 0$ is similar to that for $(0, r)$-layers with new scale indices

$$\tau = -(p + r)/2, \quad \mu = -4-p-r, \quad \lambda = -3-p-r$$
$$\kappa = -2-p(11, 12, 22), \quad \kappa = -1-p\ (13, 23), \quad \kappa = -p\ (33)$$

and relations (2.1)–(2.8) for the principal terms.

6. COMBINED PACKETS

We have seen that a packet of (0, 0)- and (1, 0)-layers in the principal part ($s = 0$) is described by Kirchhoff's relations (2.1) and Eqs (2.2)–(2.8) for displacements and stresses of the same order and in one timescale. All that is needed is to choose the appropriate averaged stiffness matrices in each layer according to (3.1) and (3.2). A group of layers of this kind is called a K-packet, or Kirchhoff-type longitudinally rigid layers.

If $(p, 0)$-layers ($p \geqslant 2$) are included in the packet it is impossible to obtain an averaged two-dimensional model and one must analyse the full three-dimensional problem.

Adding $(0, q)$ or (p, q)-layers, which satisfy the corresponding two-dimensional theories of similar packets (apart from $(q + r, q)$-layers, $r \geqslant 2$), to a K-packet requires a new analysis because the orders of the limits of the quantities for the models of each type are different.

We will investigate the important practical case where the supporting K-layers lie on the edge of the packet with (p, q)-layers inside, where $p \leqslant q$ so that the packet operates coherently in a transverse direction [6, 7, 11].

7. THE CASE OF A $(0, q)$-FILLER

$N^{\mp}$ denotes the set of indices corresponding to the numbers of the K-layers next to the face $z = z^{\mp}$, N^0 is the set of numbers of the layers of the filler ($q \geqslant 1$) and $z^- < z^+$ are the coordinates of the interfaces between the peripheral layers and the filler. In this case the asymptotically principal parts ($s = 0$) of the displacement and stress fields in the K-layers satisfy relations (2.1)–(2.4). For each group of K-layers the summation formulae in (2.3) and (2.4) must be chosen from the adjacent face and within the indices of the given group (that is, for $l, j \in N^{\pm}$ $\Sigma_{\pm}$ is taken in (2.5)). The averaged two-dimensional equations have the form (2.7), where

$$\mathbf{D}_k \equiv \mathbf{D}_k^+ + \mathbf{D}_k^- \qquad (7.1)$$

For displacements in the layers of the filler $j \in N^0$ Eqs (2.1) are again satisfied, but with a possible change in the structure and orders of the stresses. For longitudinal stresses $\sigma^j_{\alpha\beta}$ (Eqs (2.2) are preserved (with average layer stiffnesses $\gamma_{mn} = g_{mn}$), but with order $\varkappa = q - 2$; for transverse stresses we obtain

$$\sigma^j_{\alpha z} = \tau^0_\alpha, \quad \varkappa = -1 \tag{7.2}$$

$$\sigma^j_{zz} = \alpha^\pm_0 + [z - z_j \mp \textstyle\sum^0_\pm h_l](\rho_j \partial^2_t w - \nabla\tau_0), \quad \varkappa = 0 \tag{7.3}$$

$$\tau^0_\alpha = \tau^\pm_\alpha \pm [\mathbf{a}_\alpha(\mathbf{D}^\pm_1)\mathbf{u} - b_\alpha(\mathbf{D}^\pm_2)w] \tag{7.4}$$

$$\sigma^\pm_0 = \sigma^\pm + \nabla(z^\pm \boldsymbol{\tau}^\pm - z^\pm_0 \boldsymbol{\tau}_0) \mp [\rho^\pm_* \partial^2_t + b_*(\mathbf{D}^\pm_3)]w \pm \mathbf{a}_*(\mathbf{D}^\pm_2)\mathbf{u} \tag{7.5}$$

$$\Sigma^0_\pm \equiv \sum_l (l > j \quad \text{or} \quad l < j: \ l, j \in N^0)$$

where $\rho^\pm_*$ denotes density integrals in the K-layers.

In the next iteration ($s = 1$) the displacements in the K-layers can take the form (3.5), whereas the displacements in the filler satisfy Kirchhoff's relations. However, because the order of the limits changes, the stresses are completely different.

Note that the internal SSS of layers of the filler can be expressed algebraically in terms of the components (and operators) of the SSS of the supporting layers, and does not contain any additional arbitrariness for describing the boundary conditions. This is reminiscent of the behaviour of an elastic liner between absolutely rigid surfaces [19, 20].

8. A FILLER OF $(1, 1 + q)$-LAYERS

The only difference in the principal part ($s = 0$) from the previous case is that the formulae for the longitudinal stresses contain an extra term (of the same order)

$$\sigma^j_{\alpha\beta} = \boldsymbol{\chi}_{\alpha\beta}(\boldsymbol{\Gamma}_j)\mathbf{u}^0 + \boldsymbol{\delta}^j_{\alpha\beta}\boldsymbol{\tau}_0, \quad \varkappa = -2 + q$$

$$\boldsymbol{\delta}_\theta = (g_{3\theta}, g_{4\theta}, g_{5\theta})(\boldsymbol{\Delta}')^T \quad (\theta = 1, 6, 2 \leftrightarrow \alpha\beta = 11, 12, 22)$$

9. A FILLER OF THE FORM $(2, 2 + q)$

In the previous two cases, if the displacements are independent of the layer number, the longitudinal displacements in the K-layers will be piecewise-linear functions of the thickness

$$\tau = 0, \ \mu = -4, \ \lambda = -3$$
$$\mathbf{u}^\pm_0 = \mathbf{u}(\mathbf{x}, t)^\pm - z\nabla w, \quad w^0 = w(\mathbf{x}, t) \tag{9.1}$$

and depend on the group to which the layer belongs (the expression for the deflection is unchanged). Relations (2.2)–(2.4) are preserved, the only difference from Section 7 being that the characteristic longitudinal components of displacements $\mathbf{u}^\pm$ for each group of K-layers $j \in N^\pm$ are permuted.

There is a more substantial change to the system of averaged equations

$$\mathbf{a}_\alpha(\mathbf{D}^\pm_1)\mathbf{u}^\pm - b_\alpha(\mathbf{D}^\pm_2)w \pm (\tau^\pm_\alpha - \tau^0_\alpha) = 0$$

$$-\mathbf{a}_*(\mathbf{D}^+_2)\mathbf{u}^+ - \mathbf{a}_*(\mathbf{D}^-_2)\mathbf{u}^- + [\rho_* \partial^2_t + b_*(\mathbf{D}_3)]w = \sigma^+ - \sigma^- + \nabla(z^+\boldsymbol{\tau}^+ - z^-\boldsymbol{\tau}^-) \tag{9.2}$$

$$\boldsymbol{\tau}^0 = \boldsymbol{\Delta}^{-1}_0(u^+ - \mathbf{u}^-); \quad \boldsymbol{\Delta}'_0 = \sum h_j \boldsymbol{\Delta}'_j, \quad j \in N^0 \tag{9.3}$$

The first four equations ($\pm$; $\alpha = 1, 2$) in (9.2) correspond to two problems of a generalized plane stressed state in peripheral K-groups of layers; in addition to the tangential stresses on the faces of a packet, there are terms (9.3), reminiscent of a Winkler–Fuss medium [19, 21] with total compliance matrices $\boldsymbol{\Delta}'$ (similar to (3.5)).

The last equation in (9.2) corresponds to the generalized problem of the bending of the entire packets as a whole. Here the filler participates in formation of the density integral.

The deflection in the filler is governed by formula (9.1), and the longitudinal displacements and stresses will take the form

$$\mathbf{u}_j^0 = \mathbf{u}^{\pm}(\mathbf{x},t) + [(z - z_j^{\pm})\mathbf{\Delta}'_y \mp \Sigma_{\pm}^0 h_l \mathbf{\Delta}'_l]\boldsymbol{\tau}_0 - z\nabla w$$

$$\sigma_{\alpha\beta}^j = \delta_{\alpha\beta}^j \tau_0, \quad \varkappa = -3 + q$$

The transverse stresses in the filler are given by the formulae (7.2) and (7.3), but the stresses (7.4) and (7.5) on the interfaces between K-layers and the filler $z = z_0^{\mp}$ change only due to the substitution of the stress τ^0 from (9.3).

The relations for the energy of the SSS of a packet obtained by integrating Eqs (9.2) with appropriate coefficients over the thickness as follows:

$$\mathscr{E}^{\cdot} = \mathscr{P} + \mathscr{W}, \quad \mathscr{E} \equiv \mathscr{T} + \mathscr{K}, \quad (..)^{\cdot} \equiv \partial_t(..)$$

$$\mathscr{W} = \int_{\Omega} (\sigma^+ - \sigma^-)w^{\cdot} + \tau_\alpha^+(u_\alpha^-)^{\cdot} - \tau_\alpha^-(u_\alpha^-)^{\cdot} d\Omega$$

$$\mathscr{P} = \int_{\partial\Omega} Q_n^+(u_n^+)^{\cdot} + Q_\tau^+(u_\tau^+)^{\cdot} + (M_n^+ + M_n^-)\theta_n^{\cdot} + (P_n^+ + P_n^-)w_n^{\cdot} dl \tag{9.4}$$

$$\mathscr{K} = \frac{1}{2}\int_{\Omega} Q_{\alpha\beta}^+\varepsilon_{\alpha\beta}^+ + Q_{\alpha\beta}^-\varepsilon_{\alpha\beta}^- + (M_{\alpha\beta}^+ + M_{\alpha\beta}^-)\theta_{\alpha\beta} + (\mathbf{u}^+ - \mathbf{u}^-)^T \mathbf{\Delta}^{-1}(\mathbf{u}^+ - \mathbf{u}^-)d\Omega$$

$$\mathscr{T} = \frac{1}{2}\int_{\Omega} \rho_*(w^{\cdot})^2 d\Omega; \quad \varepsilon_{\alpha\beta} \equiv \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha), \quad \theta_{\alpha\beta} \equiv -\partial_{\alpha\beta}^2 w$$

Here

$$(Q_{\alpha\beta}, M_{\alpha\beta})^{\pm} \equiv \sum_{j \in N^{\pm}} \int_{z_j}^{z_{j+1}} (1,z)\sigma_{\alpha\beta}^j dz, \quad Q_{\alpha z} \equiv \sum_{j \in N^+, N^-} \int_{z_j}^{z_{j+1}} \sigma_{\alpha z}^j dz = \tag{9.5}$$

$$= \partial_\beta(M_{\alpha\beta}^+ + M_{\alpha\beta}^-) + z^+\tau_\alpha^+ - z^-\tau_\alpha^-$$

$$P_n \equiv Q_{nz} + \partial_\tau(M_\tau^+ + M_\tau^-)$$

where $\mathscr{W}$ is the power of the surface load, $\mathscr{P}$ is the integral of the power flow across the end of the packet, $\mathscr{E}$, $\mathscr{T}$, $\mathscr{K}$ are the total, kinetic and potential energies of the packet, $Q_{\alpha\beta}$, $M_{\alpha\beta}$ are the integral stresses and moments in cross-sections of the packet generated by longitudinal stresses (Q_n, Q_τ, M_n, M_τ are the projections of these stresses and moments onto the normal and tangential directions to $\partial\Omega$), $Q_{\alpha z}$ are transverse stresses (Q_{nz} is the transverse stress in a section with normal $\mathbf{n} \perp \partial\Omega$) and P_n is the transverse Kirchhoff stress.

Formulae (9.4) and (9.5) extend the classical relations of the theory of plates to the case of contrasting packets. By virtue of the energy relations (9.4) we know that the natural boundary conditions at the ends of the packets are either given displacements or angles of rotation for K-layers

$$u_n^{\pm}, \quad u_\tau^{\pm}, \quad \theta_n = -\partial_n w, \quad w \tag{9.6}$$

or integral stresses and moments

$$Q_n^{\pm}, \quad Q_\tau^{\pm}, \quad M_n, \quad P_n \tag{9.7}$$

or combination of the two. The initial conditions may be given only in the deflection: $w(0, \mathbf{x})$, $\partial_t w(0, \mathbf{x})$, $\mathbf{x} \in \Omega$.

If we now consider a $(3.3 + q)$-filler, the displacements and stresses for the peripheral K-layers keep the form (9.1), (2.2)–(2.4); but the components τ_α^0 must be removed from Eqs (9.2).

The deflection in the filler $j \in N^0$, as before, is independent of z, while the longitudinal displacements and stresses are piecewise-linear functions of z.

In order to determine these components, we now need to consider two iterations $s = 0.1$. The resulting lengthy expressions will not be given here.

10. CONCLUSION AND RESULTS

The existence of contrasting directions in anisotropic layers leads to a substantial change in the SSS of a packet. Non-classical situations arise in (p, q)-layers with $p > q$. Whereas averaged two-dimensional long-wave models containing classical kinematic relations for the principal term $s = 0$ are obtained for $(q + 1, q)$-layers, non-classical polynomial representations (as functions of the transverse coordinate) are obtained for the longitudinal displacements in the next iteration $s = 1$.

Thus, the first two iterations yield a model which is intermediate between the classical Kirchhoff description and the high-order theory of shear strains.

For $p - q \geqslant 2$ the model gives a modified three-dimensional description of the SSS.

Real fibrous composites (including high-quality unidirectional composites) correspond best to the models of (0, 0)- and (1, 0)-layers above.

A fundamentally different situation arises in the case of a packet of several groups of contrasting layers (of the non-supporting layers—soft filler type). Even in the principal approximation, the systems of averaged equations may be of large dimensions, with different equations for each group of supporting layers. Also, the boundary conditions at the ends of the packet must be assigned separately for each non-supporting group.

The basic operators in the various models are generalizations of the well-known operators of the problems of a plane stressed state and deflection, with additional Winkler terms when the filler is soft enough. The singular behaviour of the filler is similar to that of an elastic layer clamped between rigid half-spaces for which the SSS is determined algebraically, and the boundary conditions are satisfied by a complicated boundary layer.

This research was done as part of the programme of the International Association for the Promotion of Cooperation with Scientists from the Independent States of the Former Soviet Union (INTAS 96-2306).

REFERENCES

1. GOL'DENVEIZER, A. L., Derivation of an approximate theory of bending of a plate by the method of asymptotic integration of the equations of the theory of elasticity. *Prikl. Mat. Mekh.*, 1962, **26**, 668–686.
2. FRIEDRICHS, K. O. and DRESSLER, R. F., A boundary-layer theory for elastic plates. *Commun. Pure and Appl. Math.*, 1961, **14**, 1–33.
3. VASIL'YEV, V. V. *et al.*, *Composite Materials. A Handbook*. Mashinostroyeniye, Moscow, 1990.
4. CHEPIGA, V. Ye., The construction of a theory of multilayered anisotropic shells with given nominal accuracy of order h^N. *Izv. Akad. Nauk SSSR. MTT*, 1977, 4, 113–120.
5. LO, K. H., CHRISTENSEN, R. M. and WU, E. M., A high-order theory of plate deformation. P2. Laminated plates. *Trans. ASME. J. Appl. Mech.*, 1977, **44**, 669–676.
6. REDDY, J. N., A review of refined theories for laminated composite plates. *Shock. Vib. Dig.*, 1990, **22**, 7, 3–17.
7. GRIGOLYUK, E. I. and CHULKOV, P. P., *The Stability and Vibrations of Three-layered Shells*. Mashinostroyeniye, Moscow, 1973.
8. VASIL'YEV, V. V., *The Mechanics of Structures of Composite Materials*. Mashinostroyeniye, Moscow, 1988.
9. POBEDRYA, B. Ye., *The Mechanics of Composite Materials*. Izd. MGU, Moscow, 1984.
10. GRIGOLYUK, E. I. and KULIKOV, G. M., *Multilayered Reinforced Shells. The Calculation of Pneumatic Tyres*. Mashinostroyeniye, Moscow, 1988.
11. ALEKSANDROV, V. M. and SHMATKOVA, A. A., The stress–strain state of a cantilevered three-layer panel with cellular filler. *Izv. Ross. Akad. Nauk MTT*, 1995, 1, 159–167.
12. BOLOTIN, V. V., A theory of laminated plates. *Izv. Akad. Nauk SSSR. Mekhanika i Mashinostroyeniye*, 1963, 3, 65–72.
13. BOLOTIN, V. V. and NOVICHKOV, Yu. N., *The Mechanics of Multilayered Structures*. Mashinostroyeniye, Moscow, 1980.
14. GUSEIN-ZADE, M. I., The construction of a theory of the bending of laminated plates. *Prikl. Mat. Mekh.*, 1968, **32**, 232–243.
15. VOROVICH, I. I., KADOMTSEV, I. G. and USTIOV, Yu. A., The theory of plates of non-uniform thickness. *Izv. Akad. Nauk SSSR. MTT*, 1975, 3, 119–129.
16. ZORIN, I. S. and ROMASHEV, Yu. A., The stress–strain state of laminated plates of asymmetric construction. *Prikl. Mat. Mekh.*, 1988, **52**, 88–96.
17. ZAKHAROV, D. D., Asymptotic analysis of three-dimensional dynamic elastic equations for a thin multilayered anisotropic plate of arbitrary structure. *Prikl. Mat. Mekh.*, 1992, **56**, 742–749.
18. ZAKHAROV, D. D., Two-dimensional dynamic equations of a thin asymmetrically laminated elastic plate with a general form of anisotropy. *Dokl. Ross. Akad. Nauk*, 1994, **336**, 50–53.
19. AGALOVYAN, L. A. and GEVORKYAN, R. S., The asymptotic solution of mixed three-dimensional problems for two-layer anisotropic plates. *Prikl. Mat. Mekh.*, 1986, **60**, 271–278.
20. VLASOV, V. Z. and LEONT'YEV, N. N., *Beams, Plates and Shells on an Elastic Foundation*. Fizmatgiz, Moscow, 1960.

Translated by R.L.